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Computation of All Stabilizing PID Controllers for High-Order Systems with Time Delay in a Graphical Approach

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Abstract — In the present paper, the problem of computation of all stabilizing high-order time delay systems using well known and efficient proportional-integral-derivative (PID) controller is investigated in a graphical approach. An efficient approach to this important problem is presented. Based on this approach, all PID controllers that will ensure stability are determined in a (k_p, k_i, k_d) plane and then the stability boundary in a (k_i, k_d) plane for a constant value of k_p is determined and analytically described. It is shown that the stabilizing (k_i, k_d) plane consists of triangular regions. The generalized Hermite-Biehler theorem which is applicable to quasipolynomials and finite root boundaries which will be described in detail in continuation are studied to establish results for this design and also determining region of stability of designed PID controllers. Bode diagram criterion is used to show the stabilizing PID gain and phase margins. Step response is also used to show the correctness and advantage of the approach in two examples which are given to illustrate the method.

Keyword — High-order systems, PID controller, Stabilizing regions, Time-delay.

1. Introduction

Time delay is the time for system to answer a specific command after exerting an input. It can be also defined as the required time between applying change in the input and notices its effect on the system output. Delay can be seen in most of systems and its effects are considered in synthesis and analysis of systems. Delays are often causes for instability and poor performance of system and make stability analysis and controller design difficult.

Most of classical methods used for controller design cannot be used with delayed systems. This is due to the fact that the system's future behaviors depend not only on the current value of the state variables, but also some past history of the state variables.

In last decades widespread studies over time delay systems have been done. Main reasons for this development can be stated as follows [1]:

I. Delays are frequently used to simplify high-order models.
II. It can be shown that delay in difference equations can be achieved partially easy by simplification of system model.

On the other hand, PID controllers are used frequently in various engineering applications. Due to this comprehensive use of PID controllers in industrial and applications, a significant effort has been done to determine the set of all PID controllers that meet specific design goals [2]. The design was done by Ziegler and Nichols [3] for the first time and many formulas and equations have been extracted for different proposed design methods after the publication of the first design [4].

In attention to the great industrial use of this kind of controller and also the point that time delays are usually unavoidable in many mechanical and electrical systems, we can claim that even a partial improvement in PID design can be theoretically or practically effective and useful.

For this great controller many design methodologies have been studied and made. For example, in [5], the D-decomposition approach has been used to determine the stabilizing region of PID controllers. A characterization of all stabilizing PID controllers for an arbitrary plant in a computational way has been proposed in [6]. In [7], a parametric Kharitonov region for the PID controllers to guarantee stability and robustness has been proposed. A generalization of the Hermite–Biehler theorem has been used for determining the stabilizing PID controller based on the inverse Nyquist plot in [8]. The [9] computes the stabilizing PI and PID controllers to achieve gain and phase margins for processes with time delay.

Recently, the problem of finding the stabilizing PID controllers for high-order delayed systems positive or negative, has been studied in [10]. Nyquist criterion is another subject in dealing with high-order unstable delayed plants which were studied in [11].

In this paper, we present a simple and efficient approach for determining the stabilizing PID controllers for high-order time delay systems based on generalization of the Hermite-Biehler theorem is presented in a graphical approach. We also use the theorem of finite root boundaries which will be described in the next section to get better results by which the region of stability is determined and also plotted.

Bode plot is used to show the stabilizing PID gain and phase margins in the illustrative examples.

It's worthy to note that step response plot is also used to show the correctness of our proposed stabilizing method.

The organization of this paper is as follows: In Section 2, the problem is stated. We also state a definition and also two important theorems which help us with getting desired and trustworthy results. Section 3 shows stabilization using a PID controller method. In this section our approach of design is explained clearly and all stabilizing PID parameters are determined. Section 4 shows illustrating of the approach by two examples which will be analyzed by MATLAB and simulation results will show the advantages of this approach. Finally, in section 5, the main conclusions are summarized.

2. Problem Formulation And Preliminaries

PID is a combination of three controllers: proportional, derivative and integral controller. One of the main interesting features of PID controllers is their simplicity. The most general form of then is the second order system in the *s*-domain defined as follows:

$$K(s) = k_p + \frac{k_i}{s} + k_d s \ . \qquad (1)$$

We consider in this section $G(s)$ as the transfer function of high-order time delay systems defined by.

$$G(s) = \frac{N(s)}{D(s)} e^{-\tau s} = \frac{\left(\delta_m s^m + \delta_{m-1} s^{m-1} + \ldots + \delta_1 s + \delta_0\right)}{\delta_n s^n + \delta_{n-1} s^{n-1} + \ldots + \delta_1 s + \delta_0} e^{-\tau s} \qquad (2)$$

The degree of the denominator is higher than that of the numerator in this representation. Equivalently it would be strictly proper, and $n \geq m+2$ in this study.

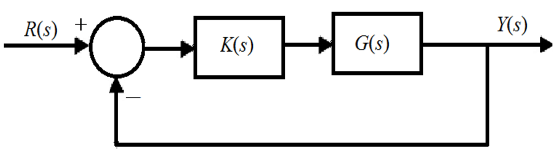

Fig.1. Unity feedback control system.

The goal in this computations is to find the regions in terms of k_p, k_i , and k_d such that the closed loop characteristic polynomial of the system in fig 1 be Hurwitz stable.

The characteristic polynomial is written as

$$\Delta(s) = 1 + K(s)G(s) \qquad (3)$$

Let $s = j\omega$,

So

$$\Delta(j\omega) = 1 + K(j\omega)G(j\omega) \qquad (4)$$

Definition 1 [8,12**]**. Let $\delta(s) = \delta_e(s^2) + s\delta_o(s^2)$ be a given real polynomial of degree n, where $\delta_e(s^2)$ and $s\delta_o(s^2)$ are the components of $\delta(s)$ made up of even and odd powers of *s*, respectively.

Definition 2 [13]. A finite root boundary in the (k_d, k_i) plane for a constant value of k_p is the locus, where characteristic function of fig 1 has a root $s = j\omega$ on the imaginary axis with a finite $\omega_\eta \in R$. If $\omega_\eta = 0$, the root boundary is called a real root boundary (RRB) and when $\omega_\eta \neq 0$, it's a complex root boundary (CRB). If the root boundary is crossed from its unstable side, the corresponding root moves from the right half plane (RHP) to the LHP.

Theorem 1 [8,12].

Let $\delta(s) = \delta_0 + \delta_1 s + \ldots + \delta_n s^n$ be a given real polynomial of degree *n*, numbers of roots of $\delta(s)$ in left-half plane and right-half plane, are denoted by l and r respectively, and $\sigma(\delta) = l - r$. Then

$$\Delta_0^{+\infty}(\angle\delta(j\omega)) = \frac{\pi}{2}\sigma(\delta),$$

where

$$\Delta_0^{+\infty}(\angle\delta(j\omega)) = \int_0^{\infty} \angle\delta(j\omega)d\omega.$$

$$\angle\delta(j\omega) \cong \theta(\omega)$$

This is called *"Generalization of the Hermite-Biehler Theorem"*.

Theorem 2 [13]

Given a delayed system with the (2) and a $f(\omega) = k_p$ fulfilling the conditions in Definition 2, the finite root boundaries in the (k_d, k_i) plane for a fixed value of k_p are the following lines

I. $k_i > 0, if\ a_0 \neq 0$ and $k_p > f_0$ (RRB),

II. $k_i < 0, if\ a_0 \neq 0$ and $k_p < f_0$ (RRB),

III. $k_i < \omega_\eta^2 k_d + g(\omega_\eta)$ for all $\omega_\eta \in \Omega^+$ (CRB)

IV. $k_i > \omega_\eta^2 k_d + g(\omega_\eta)$ for all $\omega_\eta \in \Omega^-$ (CRB)

with

$$g(\omega) = \omega\left(-f_2(\omega)\sin(\omega L) + f_1(\omega)\cos(\omega L)\right)$$

$f_0 = \lim_{\omega \to 0} f(\omega)$, which exists for $a_0 \neq 0$.

This is named as *"Finite-root-boundaries- Theorem"*.

All of these theorems and preliminaries are used in getting better results which are shown in coming examples for instance.

3. ALL STABILIZING PID CONTROLLERS

The stabilizing techniques used in this paper are based on decomposing the delayed system into real and imaginary parts and then using theorems and definitions in previous section. It can result in graphical method which is capable of ensuring closed loop stability of system.

The following four steps are proposed to find the boundaries of a PID controller that ensure the stability.

I. Decompose the frequency form of the system without delay into real and imaginary parts, and substitute them into (15)-(16) to obtain the PID parameters.
II. Analyze the system in presence of time delay, and redo the procedures.
III. Determine the PID stability boundaries of the system using their related equations in this section and theorems 1 and 2.
IV. Finally, plot the all stabilizing regions and then k_i versus k_d for a stabilizing constant value of k_p.

This approach is used in our example and the results show the advantages of it in the next section.

Now the following equations are represented in the frequency domain.

$$G(j\omega) = \operatorname{Re}(G(j\omega)) + j\operatorname{Im}(G(j\omega)) \qquad (5)$$

$$\Delta(j\omega) = 1 + \left(k_p + j(k_d\omega - \frac{k_i}{\omega})\right)\left(\operatorname{Re}(G(j\omega)) + j\operatorname{Im}(G(j\omega))\right) \qquad (6)$$

By expanding $\Delta(j\omega)$ into real and imaginary parts, (6) can be written as

$$\Delta(j\omega) = R_\Delta(j\omega) + jI_\Delta(j\omega) \qquad (7)$$

where

$$R_\Delta(j\omega) = (\omega\operatorname{Re}(G(j\omega)))k_p + (\operatorname{Im}(G(j\omega)))k_i - ((\operatorname{Im}(G(j\omega)))\omega^2)k_d + \omega \qquad (8)$$

$$I_\Delta(j\omega) = (\omega\operatorname{Im}(G(j\omega)))k_p - (\operatorname{Re}(G(j\omega)))k_i + (\omega^2\operatorname{Re}(G(j\omega)))k_d \qquad (9)$$

Now all stabilizing regions can be determined like what is shown in "figs" 2 and 5. Note that we complete the calculations here just for a constant value of k_d in the (k_p, k_i) plane, for brevity. The other cases are obtained similarly. Of course they can be seen in our example results completely. Looking through the frequency-domain form of PID controller in (6), it is clear that k_i and k_d are at the imaginary part and they would be depend on each other. Of course this point is clear in coming equations and using theorems 1 and 2 help us in determining stabilizing regions.

By putting (8)-(9) equal to zero, we have

$$\begin{cases} R_\Delta(j\omega) = 0 \\ I_\Delta(j\omega) = 0, \end{cases} \qquad (10)$$

which results in

$$k_p(\omega) = -\frac{\operatorname{Re}G(j\omega)}{\omega|G(j\omega)|^2} \qquad (11)$$

and

$$k_i(\omega) = \omega^2 k_d - \frac{\omega(\operatorname{Im}G(j\omega))}{|G(j\omega)|^2} \qquad (12)$$

where

$$|G(j\omega)|^2 = \left([\operatorname{Re}G(j\omega)]^2 + [\operatorname{Im}G(j\omega)]^2\right) \qquad (13)$$

It can be written that

$$\begin{aligned} \operatorname{Re}G(j\omega) &= |G(j\omega)|\cos(\theta(\omega)), \\ \operatorname{Im}G(j\omega) &= |G(j\omega)|\sin(\theta(\omega)) \end{aligned} \qquad (14)$$

where $\theta(\omega) = \angle G(j\omega)$.

Therefore, (11) ad (12) can be simplified as

$$k_p(\omega) = -\frac{\cos(\theta(\omega))}{\omega|G(j\omega)|} \qquad (15)$$

and

$$k_i(\omega) = \omega^2 k_d - \frac{\omega(\sin(\theta(\omega)))}{|G(j\omega)|} \qquad (16)$$

Although it's somewhat similar to the way used in [9], but the final equations in (15) and (16) are main concerns here which have not been considered in [9]. Based on these equations, it can be claimed that the k_p only affects the odd frequencies of the roots of (5), whereas the even frequencies of roots of (5) are influenced by k_i and k_d.

Now all the stabilizing boundary for (k_p, k_i, k_d) is determined first and then k_i versus k_d is plotted based on theorem 2 for a constant value of k_p in its stabilizing boundary. Bode plot can be used for determining stabilizing gain and phase margins in this condition.

According to Theorem 1, when designed PID controller makes system stable, numbers of roots of $\delta(s)$ in left-half plane are equal with degree real of polynomial $(l = n)$ and there be no number of roots of $\delta(s)$ in right-half plane $(r = 0)$.

We also should consider the roots of $G(s)$ which may have some roots in right-half plane which we call them

r_D and also some roots in left-half plane which we call them l_D, in this paper. By simplifying $G(s)$ into (15)-(16), according to theorem 1 there should be at least $n+r_D-l_D$ roots in left-half plane. Now, Stability conditions and then the stability regions can be obtained. This approach is illustrated by examples in the next section.

4. Illustrative Examples

Example 1:

In this example we consider a third order unstable transfer function time delay system

$$G(s)=\frac{1}{(s+1)^3}e^{-0.5s}$$

which has been analyzed in [5].

At first the region of stability of k_p is determined. Using Theorems and explanations in the previous sections, system is unstable in the intervals of $(-\infty \;\; -0.852)$ and $(5.37 \;\; +\infty)$. As a great result, the possible stabilizing region of k_p is the interval $(-0.852 \;\; 5.37)$.

Therefore, all stabilizing regions for the PID controller are shown in fig 2.

Now we choose a fixed k_p in this region and then determine the stabilizing region of k_i and k_d and this region is plotted and shown in fig 3.

Now after determining the stabilizing boundary of different controllers of PID, the step response of the closed loop system is plotted for a special case, $k_p=1,\ k_d=1.186,\ k_i=3.189$. From fig 4, we can see that the closed-loop system is stable. According to fig 5, stabilizing phase margin is obtained 49.5° at frequency $0.468\ \frac{\text{rad}}{\text{sec}}$, which can be concluded as a great result. Gain margin is also $7.35\,dB$ at frequency $2.36\ \frac{\text{rad}}{\text{sec}}$ which is another good factor for this design.

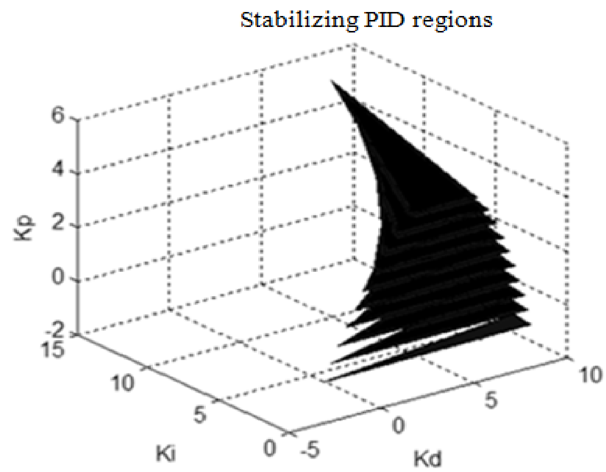

Fig.2. All stabilizing boundaries of the PID controller.

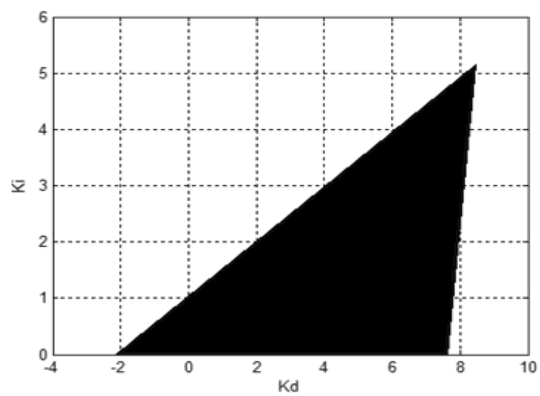

Fig.3. Stability boundaries in the (k_i, k_d) plane for $k_p=1$.

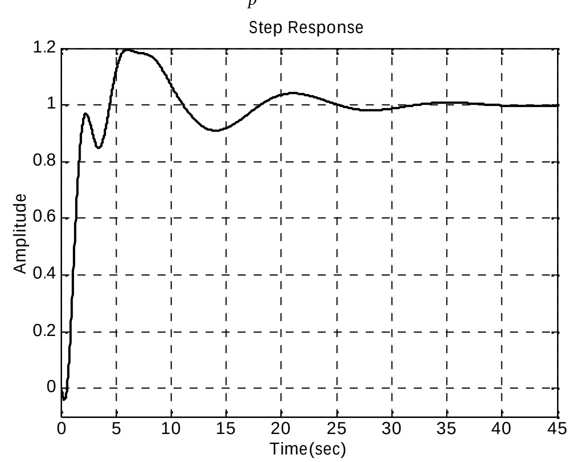

Fig.4. Closed-loop step response for system considered at example 1.

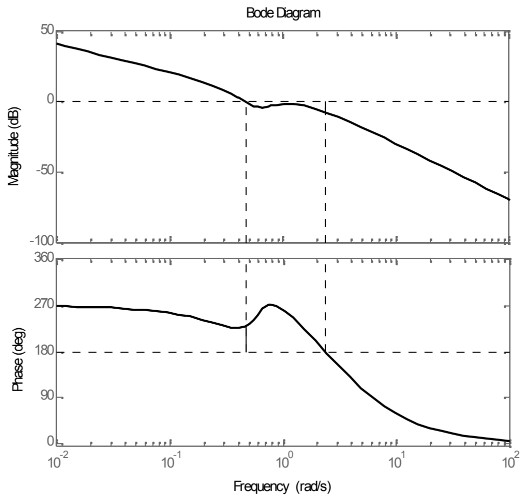

Fig.5. Bode diagram of the system at example 1.

Example 2:

In this example we consider one form of high-order transfer function which has been considered in [14], as follows:

$$G(s)=\frac{1}{(4s-1)(0.2s+1)^3}e^{-0.1s}$$

Again we want to find all stabilizing PID controllers so that the closed loop system would be stable.

Using (15), theorems 1 and 2 we obtain the stabilizing interval of k_p. Therefore, system is unstable in the k_p-intervals of $(-\infty \quad 1.12)$ and $(19.2 \quad +\infty)$. As a great result, the possible stabilizing region of k_p is the interval $(1.12 \quad 19.2\ )$.

In other words, the imaginary part $\delta(s)$ has only simple real roots if and only if $k_p \in (-0.852 \quad 5.37\ )$.

All stabilizing regions for the PID controller are shown in fig 6.

Now we choose $k_p = 5$ which is in this region and then the stabilizing region of k_i and k_d is determined and this region is plotted and shown in fig 7.

Here in this special case, the PID controller parameters and transfer function are

$$k_p = 5,\ k_d = 6.25,\ k_i = 3.84$$

$$K(j\omega) = 2 + j(3.1314\omega - \frac{1.814}{\omega})$$

It can be written that the PID controllers that ensure stability in the (k_i, k_d) plane for a constant value of k_p have been determined in a well defined set and region. In this example, we have considered $k_p = 5$ which is in the stability boundary obtained in previous computations.

By determining the possible stabilizing region, we plot the step response of the closed loop system in fig 8 to show the stability.

Bode diagram for closed-loop system is shown in fig 9 which illustrates that stabilizing gain and phase margin are achieved and the design procedure and evaluation are ended.

According to this fig, we can see that desired phase margin (54.3°) at frequency $2.79\ \frac{\text{rad}}{\text{sec}}$, has been achieved. Gain margin is also $5.89\ dB$ at frequency $5.24\ \frac{\text{rad}}{\text{sec}}$.

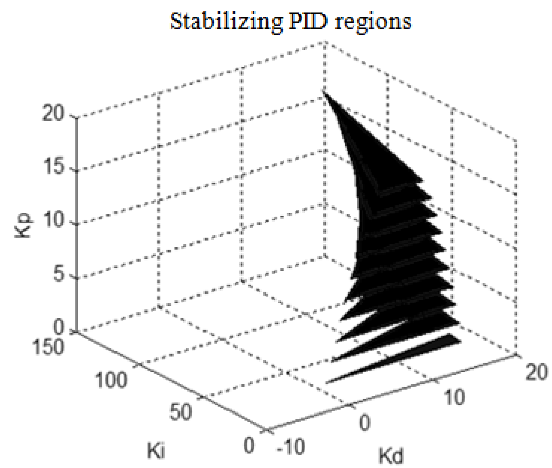

Fig.6. All stabilizing regions for the PID controller.

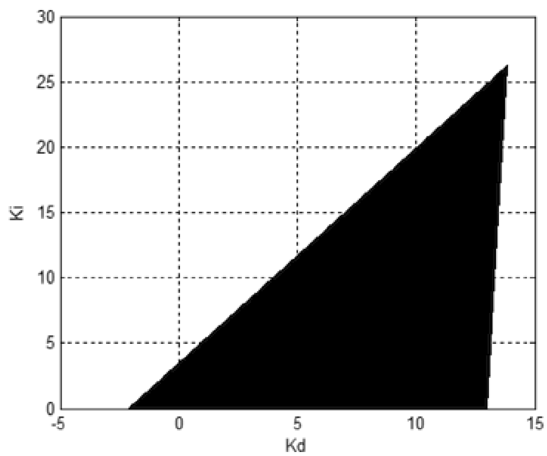

Fig.7. Stability boundaries in the (k_i, k_d) plane for $k_p = 5$.

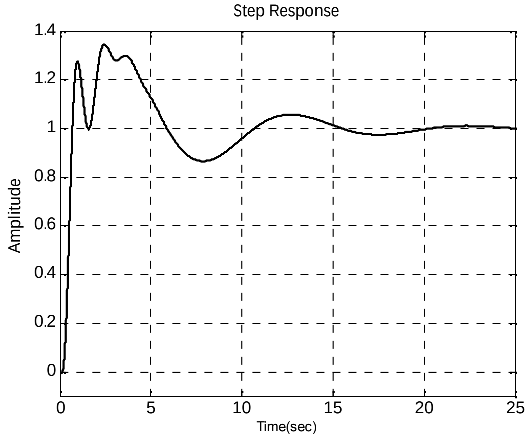

Fig.8. Closed-loop step response for example 2.

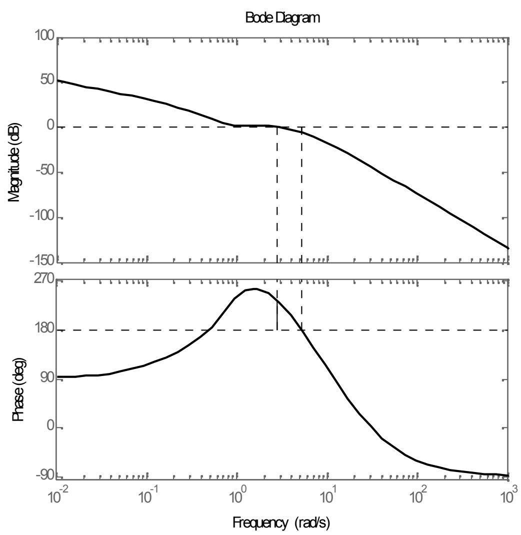

Fig.9. Bode diagram for closed-loop system of second example.

5. CONCLUSION

In this paper, we presented a graphical approach to design all stabilizing PID controllers for high-order systems with time delays. It provides an efficient and straightforward method for stabilization of PID controller. We could obtain all stabilizing PID controller in (k_p, k_i, k_d) plane and also the stabilizing regions of (k_i, k_d) for a fixed value of k_p, as well. Avoiding complex mathematical derivations and good quality and efficiency are some advantages of this approach. Numerical examples with time delay were presented to demonstrate the effectiveness of this method.

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