Current Trends in Technology and Science
Volume : 1, Issue : 2 , (Sept.-2012) ISSN : 2279-0535

Compactness in Countable Fuzzy Topological Space

Apu Kumar Saha
Assistant Professor, National Institute of Technology, Agartala, Email: apusaha_nita@yahoo.co.in

Debasish Bhattacharya
Associate Professor, National Institute of Technology, Agartala, Email: bhattacharyad_nita2007@yahoo.co.in

***Abstract* – A generalized fuzzy topological space called countable fuzzy topological space has already been introduced by the authors. The generalization has been performed by relaxing the criterion of preservation of arbitrary supremum of fuzzy topology to countable supremum. In this paper the notion of fuzzy compactness called c-compactness has been initiated and various properties are studied. Other related concepts like Lindelöf property, countable compactness are defined and studied in the countable fuzzy topological space.**

***Keywords* – c-compact space, c-Lindelöf space, countable fuzzy topology, Fuzzy topology.**

1. Introduction and Preliminaries

Soon after the historic paper of Zadeh [26], many mathematicians started to use the fuzzy sets in various branches of pure and applied Mathematics. One such application is to generalize the different topological concepts in fuzzy environment. In 1968, C. L. Chang [6] first defined fuzzy topological space using fuzzy sets and later on in 1976, R. Lowen [16], presented an alternative and more general definition of fuzzy topological space. Again, some generalizations of fuzzy topological space viz. smooth topological space [21], fuzzy supra topological space [1], fuzzy minimal space [2], fuzzy infy topological space [22] are also in literature. In [4], [23], the authors introduced the concept of countable fuzzy topological space as another generalization of fuzzy topological space and studied some properties of it. *A collection of fuzzy sets $\mathscr{F}$ on a universe X forms a countable fuzzy topology if in the definition of a fuzzy topology; the condition of arbitrary supremum is relaxed to countable supremum.* The countable fuzzy topological space is generalized enough to include fuzzy topological space. At the same time the relaxation of the stringent condition of preservation of arbitrary suprema to countable suprema opens up the possibility of having new properties which are absent in fuzzy topological space. For example, the only if part of theorem 2.6 is valid for compactness in countable fuzzy topological space but we believe that the same is not valid in fuzzy topological space. Again, this generalization of fuzzy topological space is stronger than that of fuzzy minimal space. Further, we have seen in [5],[2],[8],[9] that fuzzy topological space and fuzzy minimal space may be relevant to quantum particle physics in connection with string theory and ε^{∞} theory, so we believe that this new generalized fuzzy space will also be relevant for above studies.

In this paper, the concept of compactness in countable fuzzy topological space has been introduced and studied. The motivation of the paper is to explore whether the properties of fuzzy compactness in fuzzy minimal topological structure obtained by Alimohammady and Roohi [3] are still preserved under the countable fuzzy topological space being considered here, where compactness is defined in the usual way. The results obtained here actually generalize almost all the properties of compactness in fuzzy minimal space of [3]. This justifies the importance of present study.

Before proceeding further, we present some of the concepts and results obtained in [23] for ready reference which will be used in the study followed.

Definition 1.1: A family $\mathcal{C}$ of fuzzy sets in X is said to form a countable fuzzy topology if

i) $r\mathbf{1}_X \in \mathcal{C}$ for any $r \in I$.
ii) For countable family $\{\lambda_i : i \in N\}$ of fuzzy subsets of $\mathcal{C}$, $\vee_i \lambda_i \in \mathcal{C}$.
iii) For any two fuzzy subset λ and μ of $\mathcal{C}$, $\lambda \wedge \mu \in \mathcal{C}$.

The space (X, $\mathcal{C}$) is called countable fuzzy topological space. Every member of $\mathcal{C}$ is called c-open set of X and complement of a c-open set is called c-closed set.

Definition 1.2: We set the definition of interior and closure of a fuzzy set λ in countable fuzzy topology denoted by c-Int(λ) and c-Cl(λ) respectively as follows:

c-Int(λ) = $\vee\{\delta: \delta \le \lambda, \delta \in \mathcal{C}\}$ and c-Cl(λ) = $\wedge\{\gamma: \lambda \le \gamma, (1-\gamma) \in \mathcal{C}\}$.

Remark 1.3: It should be noted here that in a countable fuzzy topological space, arbitrary union of c-open sets may not be c-open [4]. Thus c-interior of a fuzzy set in (X, $\mathcal{C}$) may not be c-open and dually c-closure of a fuzzy set in (X, $\mathcal{C}$) may not be c-closed.

Proposition 1.4: For any two fuzzy sets λ and μ

i) c-Int(λ) $\le \lambda$ and c-Int(λ) = λ, if λ is a c-open set. Specially c-Int($r1_X$) = $r1_X$ for all $r \in I$.
ii) $\lambda \le$ c-Cl(λ) and λ = c-Cl(λ), if λ is a fuzzy c-closed set. Specially c-Cl($r1_X$) = $r1_X$ for all $r \in I$.
iii) c-Int(λ) $\le$ c-Int(μ) and c-Cl(λ) $\le$ c-Cl(μ), if $\lambda \le \mu$.
iv) c-Int($\lambda \wedge \mu$) = c-Int(λ)$\wedge$ c-Int(μ) and c-Int(λ)$\vee$ c-Int(μ) $\le$ c-Int($\lambda \vee \mu$).
v) c-Cl($\lambda \vee \mu$) = c-Cl(λ)$\vee$ c-Cl(μ) and c-Cl($\lambda \wedge \mu$) $\le$ c-Cl(λ)$\wedge$ c-Cl(μ).
vi) c-Int(c-Int(λ)) = c-Int(λ) and c-Cl(c-Cl(μ)) = c-Cl(μ).
vii) 1– c-Cl(λ) = c-Int(1– λ) and 1– c-Int(λ) = c-Cl(1– λ).

Definition 1.5: Let (X, $\mathcal{C}$) and (X, $\mathcal{D}$) be two countable fuzzy topological spaces. Then a fuzzy function f: (X, $\mathcal{C}$)

$\to$ (X, $\mathcal{D}$) is said to be countable fuzzy continuous (briefly fuzzy c-continuous) if $f^{-1}(\lambda)\in\mathcal{C}$ for any $\lambda\in\mathcal{D}$.

Theorem 1.6: Consider the following properties for a fuzzy function f: (X, $\mathcal{C}$)$\to$ (Y, $\mathcal{D}$) between two countable fuzzy topological spaces.

(a) f is fuzzy c-continuous function.
(b) $f^{-1}(\delta)$ is a fuzzy c-closed set in (X, $\mathcal{C}$) for each fuzzy c-closed set $\delta\in$(Y, $\mathcal{D}$).
(c) c-Cl($f^{-1}(\delta)$) $\le f^{-1}$(c-Cl(δ)) for each $\delta\in I^Y$.
(d) f(c-Cl(λ)) $\le$ c-Cl(f(λ)) for any $\lambda\in I^X$.
(e) f^{-1}(c-Int(δ)) $\le$ c-Int($f^{-1}(\delta)$)for each $\delta\in I^Y$.

Then (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e).

2. Fuzzy c-compact Space

After the discovery of fuzzy sets, several authors have generalized different topological concepts to fuzzy topological space. In this regard, the concept of compactness and some of its weaker and stronger forms occupies a very important place in fuzzy topology. For example we refer to [3]-[7].

For the study ahead, once again we recall the definition of countable fuzzy topology as in [4].

Definition 2.1: A family $\mathcal{C}$ of fuzzy sets in X is said to form a countable fuzzy topology if

i) $0\mathbf{1}_X$, $\mathbf{1}_X\in\mathcal{C}$.
ii) For countable family {λ_i: i$\in$N} of fuzzy subsets of $\mathcal{C}$, $\bigvee_i \lambda_i \in\mathcal{C}$.
iii) For any two fuzzy subset λ and μ of $\mathcal{C}$, $\lambda\wedge\mu\in\mathcal{C}$.

The space (X, $\mathcal{C}$) is called countable fuzzy topological space.

It can be easily verified that the properties of countable fuzzy topology already studied in [23] which are mentioned in section-1 are also valid with the present setting.

Definition 2.2: suppose (X, $\mathcal{C}$) is a countable fuzzy topological space and $\mathcal{B}$ = {λ_j: j$\in$J} is a family of fuzzy sets in X. The family $\mathcal{B}$ is called a fuzzy cover of X if $\bigvee_j \lambda_j = \mathbf{1}_X$. Also $\mathcal{B}$ is called a fuzzy cover of a fuzzy set δ in X if $\delta \le \bigvee_j \lambda_j$. It is a fuzzy c-open cover if each λ_j is fuzzy c-open. A fuzzy subcover of $\mathcal{B}$ is a subfamily of $\mathcal{B}$ which is also a fuzzy cover.

Definition 2.3: Consider (X, $\mathcal{C}$) is a countable fuzzy topological space. A fuzzy set β in X is said to be fuzzy c-compact if every fuzzy c-open cover $\mathcal{B}$ = {λ_j: j$\in$J} of β has a finite fuzzy c-open subcover. Also Y $\subseteq$ X is called is called fuzzy c-compact if $\mathbf{1}_Y$ is a fuzzy c-compact set.

Remark 2.4: It may be noticed that, the restriction of countable fuzzy topological space requiring $0\mathbf{1}_X$, and $\mathbf{1}_X$ only to be members of topology is necessary here, because $\mathbf{1}_X = \bigvee_{r\in(0,1)} \mathbf{r1}_X$ but this identity is not valid for finite choices r$\in$(0, 1), which implies that if we take the definition of countable fuzzy topological space as in [23] incorporating all constant functions in the topology then there is no fuzzy c-compact space.

Definition 2.5: A family {λ_j: j$\in$J} of fuzzy sets in X has finite intersection property if each finite subfamily of {λ_j: j$\in$J} has non-empty intersection (infimum), i.e. $\bigwedge \lambda_k \ne \mathbf{01}_X$, for k$\in$K, where K is a finite subset of J.

Theorem 2.6: A countable fuzzy topological space (X, $\mathcal{C}$) is fuzzy c-compact if and only if $\bigwedge_j \lambda_j \ne \mathbf{01}_X$ for any family {λ_j: j$\in$J} of fuzzy c-closed sets in X which has the finite intersection property.

Proof: Let the space (X, $\mathcal{C}$) be fuzzy c-compact and suppose {λ_j: j$\in$J} is a family of fuzzy c-closed subsets of X having finite intersection property. If possible let $\bigwedge_j\lambda_j = \mathbf{01}_X$. Then $\bigvee_j (1-\lambda_j) = \mathbf{1}_X$ which implies $\bigvee_{i=1}^{n} (1-\lambda_{j_i}) = \mathbf{1}_X$, which again implies $\bigwedge_{i=1}^{n} \lambda_{j_i} = \mathbf{01}_X$ which is a contradiction. Hence, $\bigwedge_j \lambda_j \ne \mathbf{01}_X$.

Conversely, suppose {μ_j: j$\in$J} is a fuzzy open cover of X and if possible, let $\bigvee_{i=1}^{n} \mu_{j_i} \ne \mathbf{1}_X$ for any choice $j_1, j_2,\ldots,j_n\in$J. Then, we have $\bigwedge_{i=1}^{n}(1-\mu_{j_i}) \ne \mathbf{01}_X$. Now, by our assumption $\bigwedge_j(1-\mu_j) \ne \mathbf{01}_X$, i.e. $\bigvee_j \mu_j \ne \mathbf{1}_X$, which is a contradiction and thus $\bigvee_{i=1}^{n} \mu_{j_i} = \mathbf{1}_X$ for a infinite subcollection {μ_{j_i}: i = 1,2,…, n} of {μ_j: j$\in$J}. So {μ_j: j$\in$J} has a finite subcover and hence the space is compact.

Now we obtain a characterization of fuzzy c-compactness using filter of c-closed set. With this aim, we recall the important concept of filter first for the sake of thoroughness.

From set theory, we know that, if a family $\mathcal{L} \subset \mathcal{P}$(X) be closed under finite intersection, then a subcollection $\mathcal{F}$ of $\mathcal{L}$ is called a filter if (i) $\phi\notin\mathcal{F}$ (ii) A, B$\in\mathcal{F}$ implies A$\cap$B$\in\mathcal{F}$ (iii) if A$\in\mathcal{F}$ and B$\in\mathcal{L}$ be such that B $\supset$ A then B$\in\mathcal{F}$. Again, we recall that for any subfamily $\mathcal{B}$ of $\mathcal{P}$(X) having the property that the intersection of any finite number of members of $\mathcal{B}$ is non-empty, i.e. having finite intersection property, there is a unique smallest filter $\mathcal{F}$ containing $\mathcal{B}$ which is called the filter generated by $\mathcal{B}$ (subbase). If $\mathcal{B}$ is also closed under finite intersection then $\mathcal{F}$ takes the simple form $\mathcal{F}$ = {A $\subseteq$ X: $\exists$ B$\in\mathcal{B}$, B $\subseteq$ A} and $\mathcal{B}$ is called a filter base for $\mathcal{F}$ [7].

Theorem 2.7: A countable fuzzy topological space (X, $\mathcal{C}$) is fuzzy c-compact if and only if $\bigwedge_{\lambda\in\mathcal{B}}$ c-Cl(λ) $\ne \mathbf{01}_X$ for every fuzzy filter subbase $\mathcal{B}$ in X.

Proof: Let us assume that $\bigwedge_{\lambda\in\mathcal{B}}$ c-Cl(λ) $\ne \mathbf{01}_X$ for every fuzzy filter subbase $\mathcal{B}$ in X. Suppose {λ_j: j$\in$J} is a family of fuzzy c-closed sets which satisfy the finite intersection property. It can be easily seen that {λ_j: j$\in$J} is a fuzzy filter subbase for X. Then by assumption $\bigwedge_j \lambda_j = \bigwedge_j$ c-Cl(λ_j) $\ne \mathbf{01}_X$, and thus by theorem 2.6, X is fuzzy c-compact.

Conversely, let X is fuzzy c-compact and there is a fuzzy filter subbase {μ_j: j$\in$J} such that $\bigwedge_j$ c-Cl(μ_j) = $\mathbf{01}_X$. Now, from the definition of c-closure, we can write c-Cl(μ_j) = $\bigwedge_k \mu_{(j,k)}$, where $\mu_{(j,k)}$'s are fuzzy c-closed subsets and $\mu_j \le \mu_{(j,k)}$ for every k in the index set K. Then from the assumption $\bigwedge_j \bigwedge_k \mu_{(j,k)} = \mathbf{01}_X$ and thus $\bigvee_j \bigvee_k (1-\mu_{(j,k)}) = \mathbf{1}_X$. Again, since X is fuzzy c-compact, we have $\bigvee_{i=1}^{n} (1-\mu_{(j_i,k_i)}) = \mathbf{1}_X$ for a finite subcollection {$(1-\mu_{(j_i,k_i)})$: i

Current Trends in Technology and Science
Volume : 1, Issue : 2 , (Sept.-2012) ISSN : 2279-0535

= 1,2,... n} of {(1−$\mu_{(j,k)}$): j∈J and k∈K} and so $\wedge_{i=1}^{n}\mu_{(j_i,k_i)}$ =$\mathbf{0_X}$.

Since c-Cl(μ_{j_i}) = $\wedge_k \mu_{(j_i,k)} \le \mu_{(j_i,k_i)}$ for each i = i,2, ..., n. So $\wedge_{i=1}^{n}$ c-Cl(μ_{j_i}) = $\mathbf{0_X}$. Therefore, {μ_j: j∈J} does not have the finite intersection property which is a contradiction. Hence, $\wedge_j$ c-Cl(μ_j) ≠ $\mathbf{0_X}$.

Theorem 2.8: Suppose f: (X, 𝒞)→(Y, 𝒟) is fuzzy c-continuous. Then, if λ is fuzzy c-compact in X then f(λ) is also fuzzy c-compact.

Proof: Let {δ_j: j∈J} be a family of fuzzy c-open sets in Y such that f(λ) ≤ $\vee_j \delta_j$. Again, we have

$\lambda \le f^{-1}(f(\lambda)) \le f^{-1}(\vee_j \delta_j) = \vee_j(f^{-1}(\delta_j))$, j∈J.

Now, since f is fuzzy c-continuous and λ is fuzzy c-compact, so there exists $j_1, j_2, ..., j_n$∈J such that $\lambda \le \vee_{i=1}^{n}(f^{-1}(\delta_{j_i}))$; i.e. $\lambda \le f^{-1}(\vee_{i=1}^{n}(\delta_{j_i}))$. Consequently, $f(\lambda) \le f(f^{-1}(\vee_{i=1}^{n}(\delta_{j_i}))) \le \vee_{i=1}^{n}(\delta_{j_i})$, and hence f(λ) is fuzzy c-compact.

Theorem 2.9: Suppose (X, 𝒞) is a fuzzy c-compact space and f: (X, 𝒞)→(Y, 𝒟) is a surjective fuzzy c-continuous. Then Y is fuzzy c-compact.

Proof: Since f is onto, we have f($\mathbf{1_X}$) = $\mathbf{1_Y}$. Now applying theorem 2.8, we get the required result.

Definition 2.10: A fuzzy function f: (X, 𝒞)→(Y, 𝒟) is called fuzzy c-open map if f(λ)∈𝒟 for each λ∈𝒞 for two countable fuzzy topological space (X, 𝒞) and (Y, 𝒟).

Definition 2.11: A fuzzy function f: (X, 𝒞)→(Y, 𝒟) is called fuzzy c-quasi open if f(λ)∈𝒟 implies λ∈𝒞 for two countable fuzzy topological space (X, 𝒞) and (Y, 𝒟).

Corollary 2.12: suppose f: (X, 𝒞)→(Y, 𝒟) is onto and fuzzy c-quasi open. Then Y is fuzzy c-compact whenever X is fuzzy c-compact.

Proof: It is easy to see that f is fuzzy c-continuous and onto. Now, applying theorem 2.8, we may obtain the required result.

In [23], authors introduced the concept of initial (weakest) countable fuzzy topology on a set X for a family of fuzzy functions f_j: X→(Y_j, $𝒞_j$) and also product countable fuzzy topological space for an arbitrary family {(X_j, $𝒞_j$): j∈J} of countable fuzzy topological spaces. In fact, product countable fuzzy topology on X = $\prod_{j\in J}X_j$ is the weakest countable fuzzy topology on X, denoted by 𝒞 = $\prod_{j\in J}𝒞_j$, such that for each i∈J, the canonical projection π_i: $\prod_{j\in J}X_j \to X_i$ is fuzzy c-continuous function.

Corollary 2.13: suppose {(X_j, $𝒞_j$): j∈J} is a family of countable fuzzy topological spaces. Then each (X_j, $𝒞_j$) is fuzzy c-compact if ($\prod_j X_j$, $\prod_j 𝒞_j$) is fuzzy c-compact.

Proof: Fuzzy c-compactness of (X_j, $𝒞_j$) follows from surjectivity and c-continuity of each π_j and theorem 2.9 with the assumption that ($\prod_j X_j$, $\prod_j 𝒞_j$) is fuzzy c-compact.

Remark 2.14: Consider a countable fuzzy topological space (X, 𝒞) and a fuzzy cover ℬ = {λ_j: j∈J} for X. Since $\vee_j \lambda_j = 1_X$, for any ε∈(0, 1) and any x∈X, there is j∈J such that $\lambda_j(x) \ge 1-\varepsilon$. Select one such λ_j and set $\Gamma_{j,\varepsilon}$ = {x: $\lambda_j(x) \ge 1-\varepsilon$}. For any fixed ε, {$\Gamma_{j,\varepsilon}$: j∈J} is called ε-partition of X by ℬ. If in the definition of ε-partition we take ε = 0, then this family is called 0-partition of X and each member of it is denoted by $\Gamma_{j,0}$ [25].

Theorem 2.15: A countable fuzzy topological space (X, 𝒞) is fuzzy c-compact if and only if every fuzzy c-open cover has a finite 0-partition.

Proof: Suppose the countable fuzzy topological space (X, 𝒞) is fuzzy c-compact and ℬ = {λ_j: j∈J} is a fuzzy c-open cover of X. Then ℬ has a finite subcover, say, $ℬ_0$ = {λ_i: i = 1, 2, ... ,n} and $\vee_{i=1}^{n}\lambda_i$ = $\mathbf{1_X}$. Thus for a fixed x∈X, there exists at least one λ_i, i = 1, 2, ..., n, say, λ_k such that $\lambda_k(x)$ = 1. For x∈X, we select one such λ_k and we set $\Gamma_{k,0}$ = {x: $\lambda_k(x)$ = 1}. Thus {$\Gamma_{k,0}$: k=1,2, ...,n} is a 0-partition of X by $ℬ_0$. Therefore $ℬ_0$ has a 0-partition of X. Again $ℬ_0$ is a subfamily of ℬ, and hence ℬ has a 0-partition of X.

Conversely, let every fuzzy c-open cover of X has a finite 0-partition. Then any fuzzy c-open cover ℬ = {λ_j: j∈J} has a finite 0-partition {$\Gamma_{i,0}$: i = 1, 2, ..., n}. Let λ_i be a fuzzy set in correspondence of $\Gamma_{i,0}$ mentioned in the remark 2.14 before the theorem. Then, corresponding to every $\Gamma_{i,0}$, there is a c-open fuzzy set λ_i such that $\Gamma_{i,0}$ = {x: $\lambda_i(x)$ = 1} and thus $\vee_i \lambda_i$ = $\mathbf{1_X}$ for i = 1, 2, ..., n. Therefore {λ_i: i = 1, 2, ... ,n} is a finite subfamily of ℬ which is also a fuzzy cover of X and hence (X, 𝒞) is compact.

Corollary 2.16: Suppose (X, 𝒞) is a countable fuzzy topological space. Then X is not fuzzy c-compact if there exists a fuzzy c-open cover ℬ of X with $\lambda_j(x) < 1$ for all λ_j∈ℬ.

Proof: By the first part of the theorem 2.15, for a countable fuzzy topological space (X, 𝒞) to be fuzzy c-compact it is necessary that every c-open cover ℬ of X has a finite 0-partition of X by ℬ. That is, for each x∈X, there must exists some λ_j∈ ℬ such that $\lambda_j(x)$ = 1. Thus if there exists a point x∈X such that $\lambda_j(x) < 1$ for all λ_j∈ ℬ, then the above condition is not fulfilled and (X, 𝒞) cannot be fuzzy c-compact.

3. Fuzzy Countably c-compact Space

Definition 3.1: A countable fuzzy topological space (X, 𝒞) is said to be fuzzy countably c-compact if for every countable collection {λ_n: n∈N} of fuzzy c-open sets for which $\mathbf{1_X} = \vee_{n\in N}\lambda_n$, there exists $n_1, n_2, ..., n_k$∈N such that $\mathbf{1_X} = \vee_{i=1}^{k}\lambda_{n_i}$.

Alternatively, a countable fuzzy topological space (X, 𝒞) is said to be fuzzy countably c- compact if every fuzzy countable c-open cover of X has a finite subcover.

Definition 3.2: A countable fuzzy topological space (X, 𝒞) is said to be fuzzy c-C_{II} if there is a countable subfamily ℬ of 𝒞 such that any member of 𝒞 can be expressed as the supremum of members of ℬ.

Theorem 3.3: Suppose a countable fuzzy topological space (X, 𝒞) is fuzzy c-C_{II}. Then X is fuzzy c-compact if and only if it is fuzzy countably c-compact.

Proof: The first part of the theorem is obvious. For the converse, suppose X is a fuzzy countably c-compact space, and ℱ = {λ_j: j∈J} is a family of fuzzy c-open sets with $1_X = \vee_{j\in J} \lambda_j$. Since X is fuzzy c-$C_{II}$, so there is a countable subfamily 𝓑 of 𝓒 such that $\lambda_j = \vee_{i=1}^{i_j} \delta_{j_i}$, where each $\delta_{j_i} \in$ 𝓑 and i_j may be infinity also. Again, let $𝓑_0$ = $\{\delta_{j_i}$: j∈J, 1≤ i ≤ $i_j\}$. Clearly, $𝓑_0$ ⊆ 𝓑. Then, $𝓑_0$ forms a countable fuzzy c-open cover for X. Consequently, there is a finite subcover $𝓑_1$ ⊆ $𝓑_0$ for X. But each member of $𝓑_1$ is contained in λ_j for some j∈J and so these λ_j's form a finite subcover, i.e. the cover 𝓕 has finite subcover. Hence, X is c-compact.

Theorem 3.4: A countable fuzzy topological space (X, 𝓒) is fuzzy countably c-compact if and only if every countable fuzzy c-open cover has a finite 0-partition.

Proof: It can be proved in a similar fashion as in theorem-2.15.

Corollary 3.5: Suppose (X, 𝓒) is a countable fuzzy topological space. Then X is not fuzzy countably c-compact if there exists a fuzzy countable c-open cover 𝓑 of X and a point x∈X with $\lambda_j(x) < 1$ for all λ_j∈𝓑.

Proof: It is a direct consequence of theorem 3.4.

Definition 3.6: A countable fuzzy topological space (X, 𝓒) is said to be fuzzy c-Lindelöf if every fuzzy c-open cover of X has a fuzzy countable subcover.

Now, we characterize fuzzy c-Lindelöf space in the following theorem.

Theorem 3.7: A countable fuzzy topological space (X, 𝓒) is fuzzy c-Lindelöf if and only if $\wedge_j \lambda_j \neq \mathbf{0_X}$, for any family $\{\lambda_j$: j∈J} of fuzzy c-closed sets in X, where $\wedge_{j\in K} \lambda_j \neq \mathbf{0_X}$ for any countable subset K of J.

Proof: Let (X, 𝓒) satisfies the given condition and suppose $\{\mu_j$: j∈J} of fuzzy c-open cover of X and if possible $\vee_{j\in K} \mu_j \neq \mathbf{1_X}$ for any countable subset K of J. Then, $\wedge_{j\in K} (1-\mu_j) \neq \mathbf{0_X}$. Again, from the assumption $\wedge_j(1-\mu_j) \neq \mathbf{0_X}$, i.e. $\vee_j \mu_j \neq \mathbf{1_X}$, which is a contradiction.

For the converse, let (X, 𝓒) be a fuzzy c-Lindelöf space and $\{\lambda_j$: j∈J} be a family of fuzzy c-closed sets in X having countable intersection property, i.e. $\wedge_{j\in K} \lambda_j \neq \mathbf{0_X}$ for any countable subset K of J. We are to prove that $\wedge_j \lambda_j \neq \mathbf{0_X}$. If possible, let $\wedge_j \lambda_j = \mathbf{0_X}$. Then, $\vee_j(1-\lambda_j) = \mathbf{1_X}$. By, Lindelöf property, $\vee_{j\in K}(1-\lambda_j) = \mathbf{1_X}$, i.e. $\wedge_{j\in K} \lambda_j = \mathbf{0_X}$, which is a contradiction.

Theorem 3.8: A countable fuzzy topological space (X, 𝓒) is fuzzy c-Lindelöf if and only if $\wedge_{\lambda\in 𝓑}$ c-Cl(λ) ≠ $\mathbf{0_X}$ for every family 𝓑 of fuzzy sets on X, where the intersection of each countable subfamily of 𝓑 is non-empty.

Proof: The proof of the theorem is same as theorem 2.6.

Theorem 3.9: Every fuzzy c-C_{II} space is a fuzzy c-Lindelöf space.

Proof: Suppose (X, 𝓒) is a countable fuzzy topological space having c-C_{II} property and 𝓖 = $\{\mu_j$: j∈J} is a fuzzy c-open cover of X. From the assumption there is a countable subfamily 𝓑 = $\{\delta_n$: n∈N} of 𝓒 such that each μ_j can be expressed as $\mu_j = \vee_{i=1}^{i_j} \delta_{j_i}$, where i_j may be infinity. Let $𝓑_0$ = { δ_{j_i}: j∈J, 1 ≤ i ≤ i_j}. Now, $𝓑_0$ is countable, since all the members of $𝓑_0$ are from 𝓑, a countable family and covers X. Also, since each member of $𝓑_0$ is contained in μ_j for some j∈J, hence these μ_j's form a fuzzy countable subcover of X.

Theorem 3.10: suppose (X, 𝓒) is a fuzzy c-Lindelöf space and f: (X, 𝓒)→(Y, 𝓓) is a surjective fuzzy c-continuous function. Then Y is fuzzy c-Lindelöf.

Proof: This can be proved in a similar way that of theorem 2.8 and theorem 2.9.

Theorem 3.11: A countable fuzzy topological space (X, 𝓒) is fuzzy c-Lindelöf if and only if every fuzzy c-cover of X has countable ε-partition of X for all ε∈(0, 1).

Proof: Let (X, 𝓒) be a c-Lindelöf space and 𝓖 = $\{\mu_j$: j∈J} is a fuzzy c-open cover of it. Therefore 𝓖 has a countable subcover, say, 𝓑 = $\{\mu_n$: n∈N}. Now for each ε∈(0, 1), we have an ε-partition of X by 𝓑. Since 𝓑 is countable, so this ε-partition of X is countable and also it is a ε-partition by 𝓖, because 𝓑 is a subfamily of 𝓖.

Conversely, let 𝓖 be a fuzzy c-open cover of X and 0 < ε < 1. Suppose $\{\Gamma_{i,0}$: i∈I(ε)} is a countable ε-partition of X by 𝓖, and $\Gamma_{i,\varepsilon}$ is defined by the member of $\lambda_{i,\varepsilon}$ of 𝓖. Let $\varepsilon = \frac{1}{n}$, n = 2, 3,...... . Then the family $\{\lambda_{i,\varepsilon}$: i∈I(ε), $\varepsilon = \frac{1}{n}$, n = 2, 3, ...} forms a countable fuzzy subcover of 𝓖 and hence (X, 𝓒) is fuzzy c-Lindelöf.

Conclusion: A new generalized fuzzy topological space viz. countable fuzzy topological space has been studied by the author earlier. Some properties of that space have been mentioned here and also the concept of compactness on the new space called c-compact space has been introduced and studied. Also the notion of c-Lindelöf space has been defined and various properties are examined. We hope that the newly introduced space will be a powerful tool to study the various properties of fuzzy topology and this endeavor is a small step towards that goal.

References

1) M. E. Abd. El. Monsef and A. E. Ramadan, On fuzzy supra topological spaces, Indian J. pure and appl. Math 18 (1987), 322 – 329.
2) M. Alimohammady and M. Roohi, Fuzzy minimal structure and fuzzy minimal vector spaces, Chaos, Solitons and Fractals 27 (2006), 599 – 605.
3) M. Alimohammady and M. Roohi, Compactness in fuzzy minimal spaces, Chaos, Solitons and Fractals 28 (2006), 906 – 912.
4) D. Bhattacharya and A. K. Saha, Fuzzy topological spaces induced by regular lower semi-continuous functions, Proc. Nat. Sem. On Fuzzy Math and its appl. Nov 25—26, 2006, 47—56.
5) M. Caldas and S. Jafari, θ-Compact fuzzy topological spaces, Chaos, Solitons & Fractals 25 (2005) 229 – 32.
6) C. L. Chang, Fuzzy topological space, J. Math. Anal. Appl. 24 (1968) 182 – 190.
7) J. Dugundji, Topology, Allyn ANS Bacon, 1996.
8) M. S. El Naschie, On the uncertainty of cantorian geometry and the two-slit experiment, Chaos, Solitons and Fractal 9(3) (1998) 517–529.

9) M. S. El Naschie, On the unification of heterotic strings, M theory and $\varepsilon^{(\infty)}$ theory, Chaos, Solitons and Fractals 11(14) (2000), 2397–2408.
10) R. Erturk, Some results on fuzzy compact spaces, Fuzzy Sets and Systems 70(1) (1995) 107–112.
11) J. A. Goguen, The fuzzy Tychonoff theorem, J Math. Anal. Appl. 43(1973) 734–742.
12) M. Guangwu, On countably strong fuzzy compact sets in L-fuzzy topological spaces, Fuzzy Sets and Systems 72(1) (1995) 119–123.
13) I. M.Hanafy, δ-Compactness in fuzzy topological spaces, J Fuzzy Math 5(2) (1997) 277–284.
14) I. M. Hanafy, RS-Compactness in L-fuzzy topological spaces, Kyngpook Math J 42 (2002) 417–428.
15) I. M. Hanafy, Fuzzy β-compactness and fuzzy β-closed spaces, Turk J Math 28 (2004) 281–293.
16) R.Lowen,Fuzzytopological spaces and fuzzy compactness, J. Math. Anal. Appl. 56 (1976) 621–633.
17) R. Lowen, Initial and final fuzzy topologies and the fuzzy Thychonoff theorem, J Math. Anal. Appl. 58 (1977) 11–23.
18) R. Lowen, A comparison of different compactness notions in fuzzy topological spaces. J. Math. Anal. Appl. 64 (1978) 446–54.
19) A. S. Mashhour, A. A. Allam, K. M. Abd El-Hakeim, On fuzzy semicompact spaces, Bull Fas Sci Assiut Univ 16(1) (1987) 277–85.
20) S. Nanda, Strongly compact fuzzy topological spaces. Fuzzy Sets and Systems 42 (1991) 259 – 262.
21) A. A. Ramadan, Smooth Topological Space, Fuzzy Sets and Systems, 48(3) 1992 371– 375.
22) A. K. Saha and D. Bhattacharya, n-infy Induced Fuzzy Topological Spaces, communicated, 2011.
23) A. K. Saha and D. Bhattacharya, countable fuzzy topological spaces and countable fuzzy topological vector spaces, communicated, 2011.
24) A. P. Sostak, Compactness and connectedness degrees of fuzzy topological spaces, general topology and its relations to modern analysis and algebra. Berlin: Heldermann Verlag; 1988. p. 519–32.
25) C. K. Wong, Covering Properties of Fuzzy Topological Spaces, J. Math. Anal. Appl. 43 (1973) 697 – 704.
26) L. A. Zadeh, Fuzzy sets, Inform. and Control, 8 (1965) 338 – 353.

AUTHOR'S PROFILE

First Author

Dr. Apu Kumar Saha (DOB-25/08/1972) completed his M. Sc. in Mathematics from Tripura University in the year 1998 and completed his Ph.D. from National Institute of Technology, Agartala in 2012. He has a teaching experience of more than 10 years. Presently he is working as Assistant Professor in the Department of Mathematics, National Institute of Technology Agartala, Jirania, Tripura (west), India, Pin: 799055. He is teaching several subjects in under graduate and post graduate level. His subject of teaching is Engineering Mathematics, Topology, Complex Variables, Partial Differential Equation, Fuzzy Set Theory etc. His research interest is Fuzzy Topology.

Second Author

Dr. Debasish Bhattacharjee, (DOB-28/07/1961) completed his
M. Sc. in Mathematics from Calcutta University in the year 1983 and completed his Ph.D. from Tripura University in 1998. He has a teaching experience of more than 25 years. Presently he is working as Associate Professor in the Department of Mathematics, National Institute of Technology Agartala, Jirania, Tripura (west), India, Pin: 799055.
He is a popular teacher in under graduate and post graduate level. His subject of teaching is Engineering Mathematics, Topology, Riemannian Geometry, Classical Mechanics. Besides teaching he is actively engaged in research field. His research fields of Operation Research, Fuzzy systems and Logic, Topology.